Interactive Formal Verification
3: Elementary Proof

Tjark Weber
(Slides: Lawrence C Paulson)
Computer Laboratory
University of Cambridge

Goals and Subgoals

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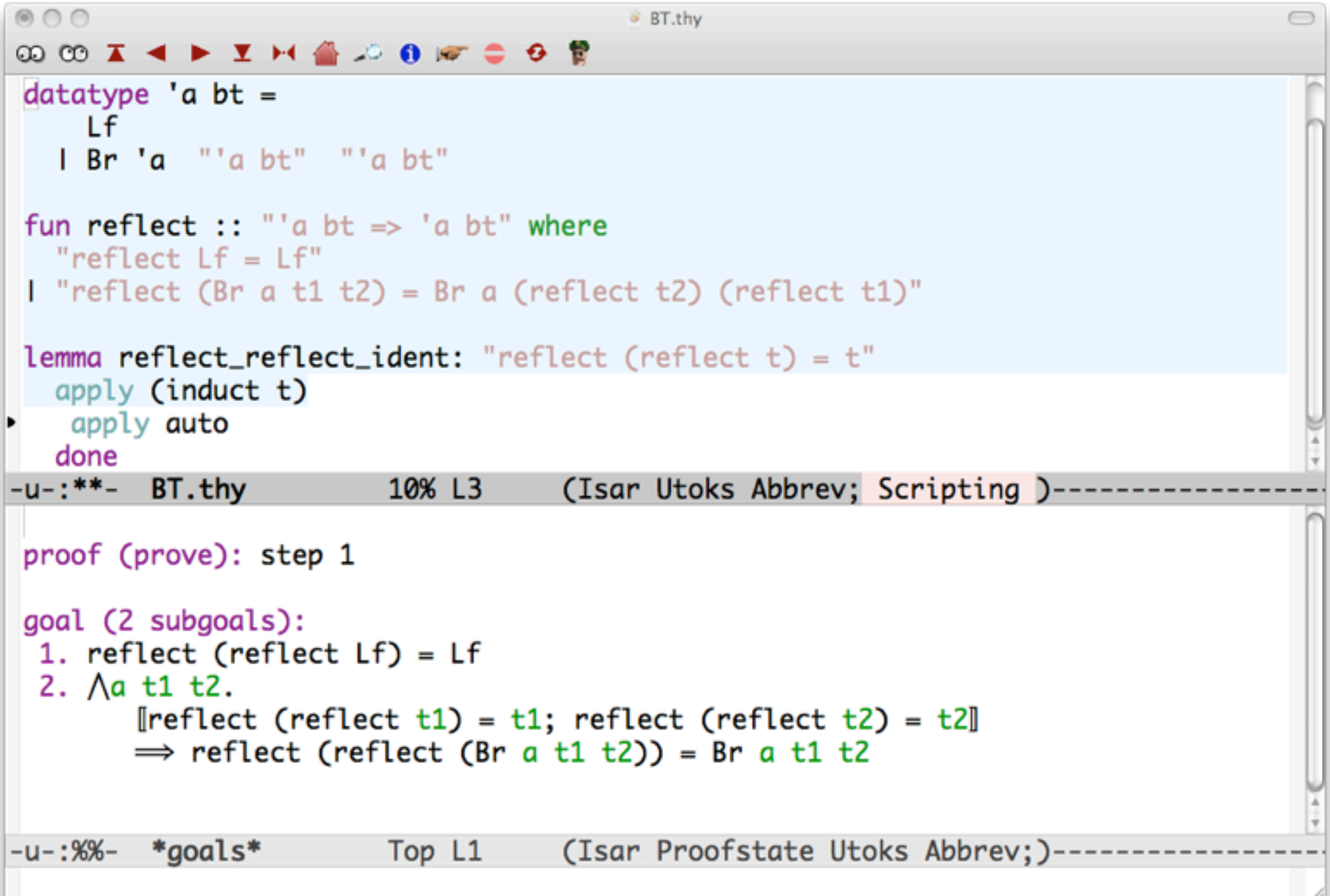
- We finish when no subgoals remain.

Structure of a Subgoal



Structure of a Subgoal

```
datatype 'a bt =
    Lf
  | Br 'a  "'a bt"  "'a bt"

fun reflect :: "'a bt => 'a bt" where
  "reflect Lf = Lf"
| "reflect (Br a t1 t2) = Br a (reflect t2) (reflect t1)"

lemma reflect_reflect_ident: "reflect (reflect t) = t"
  apply (induct t)
   apply auto
  done
```

```
-u-:**-  BT.thy        10% L3      (Isar Utoks Abbrev; Scripting )----------------
```

assumptions (two induction hypotheses)

```
 2. ⋀a t1 t2.
       ⟦reflect (reflect t1) = t1; reflect (reflect t2) = t2⟧
       ⟹ reflect (reflect (Br a t1 t2)) = Br a t1 t2
```

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-u-:%%-  *goals*       Top L1      (Isar Proofstate Utoks Abbrev;)----------------
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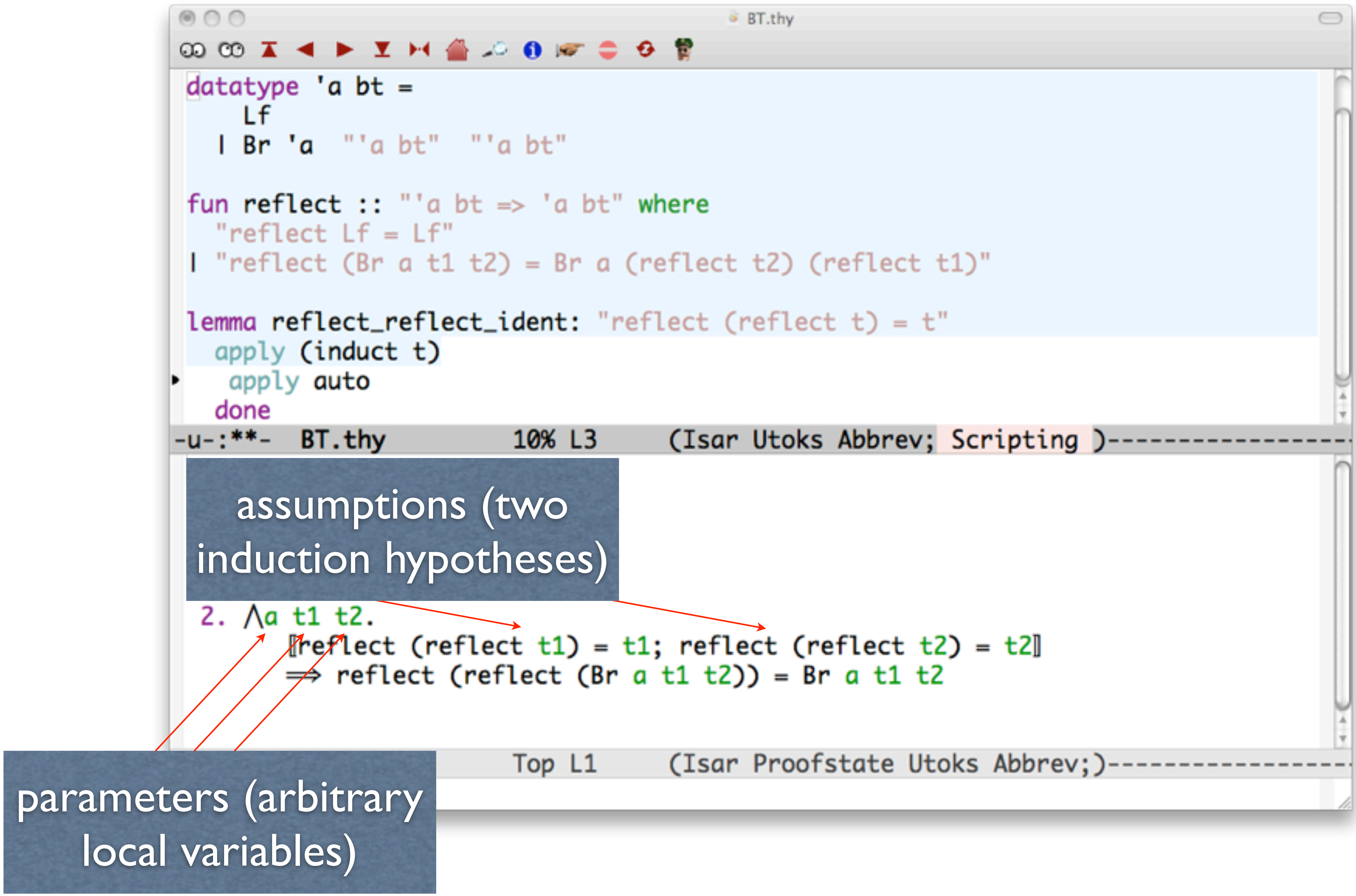

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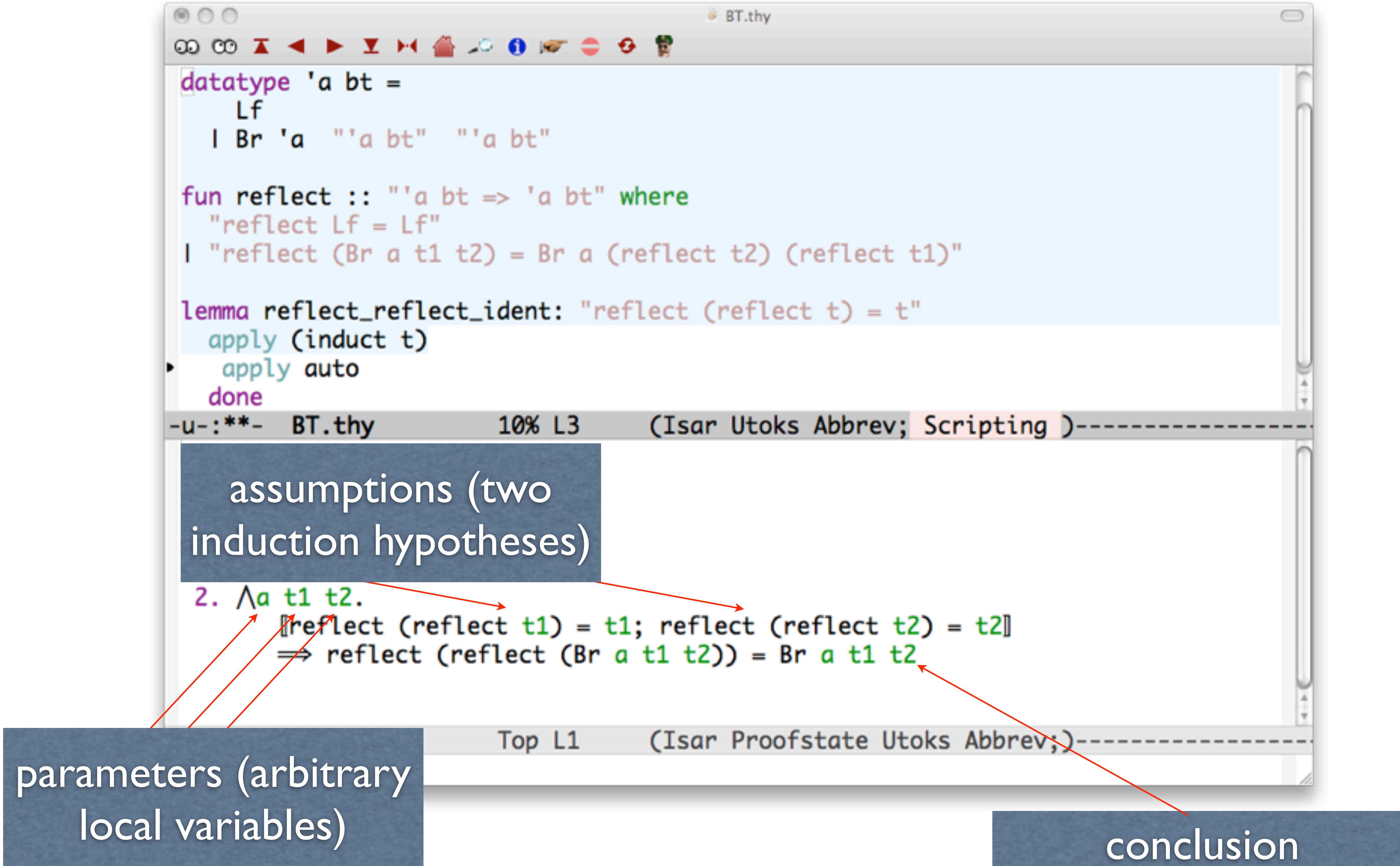

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- They can also be written with the syntax $P \leftrightarrow Q$.

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- Other case splitting can be enabled.

Conditional Rewrite Rules

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xs ≠ [] ⇒ hd (xs @ ys) = hd xs

n ≤ m ⇒ (Suc m) - n = Suc (m - n)

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- `x+y = y+x` is actually okay!
  - *Permutative rewrite rules* are applied but only if they make the term “lexicographically smaller”.

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- `auto` also applies all obvious *logical steps*
  - Splitting conjunctive goals and disjunctive assumptions
  - Performing obvious quantifier removal

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`auto` with options

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- `thm name` prints theorems.

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- Complicating issues include unusual recursions and auxiliary variables.

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- Naturally, the first thing to try is "`auto`", but much more is possible.

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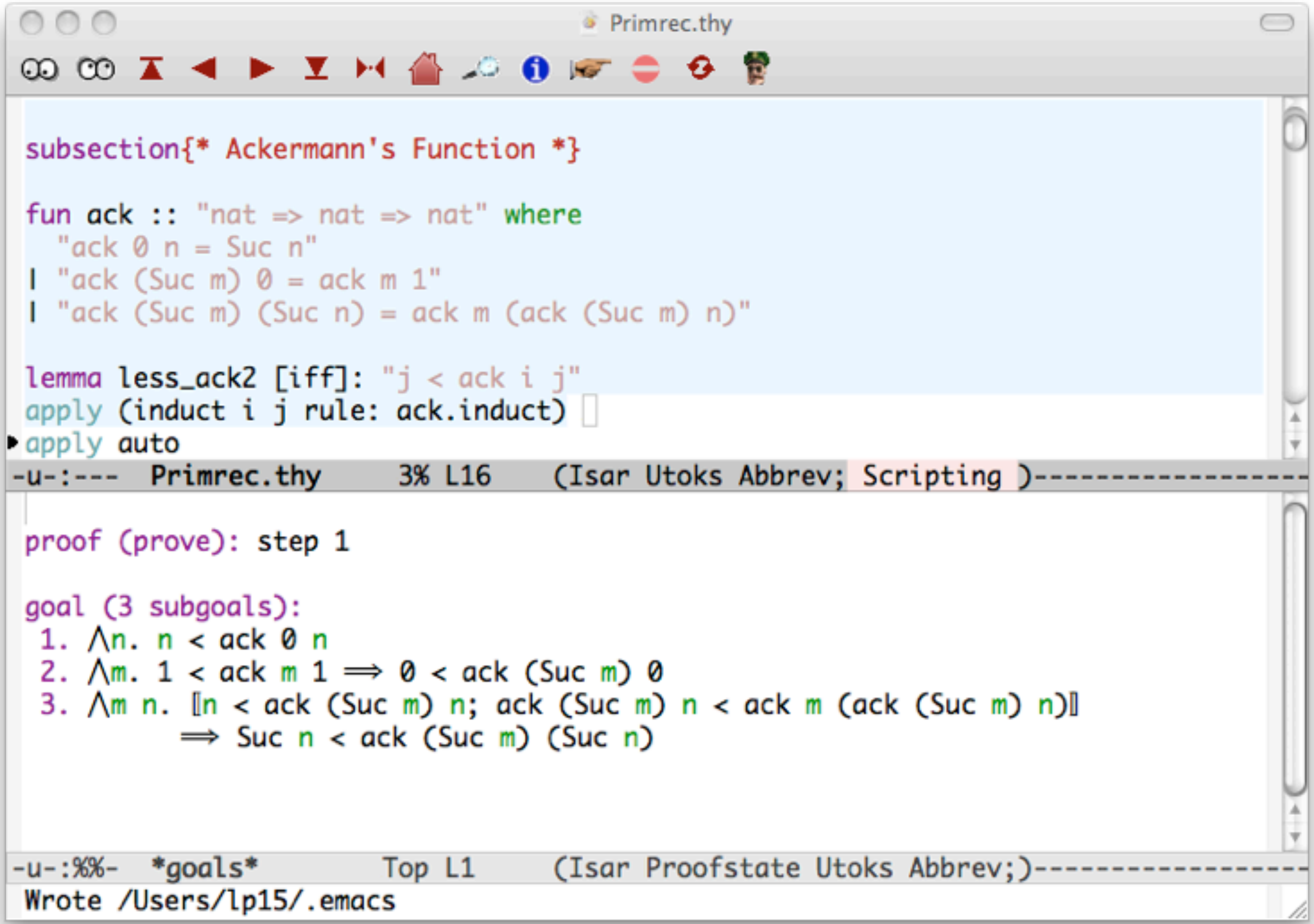
- (A different user interface: `isabelle jedit`)

Proof General Tools



Proof General Tools

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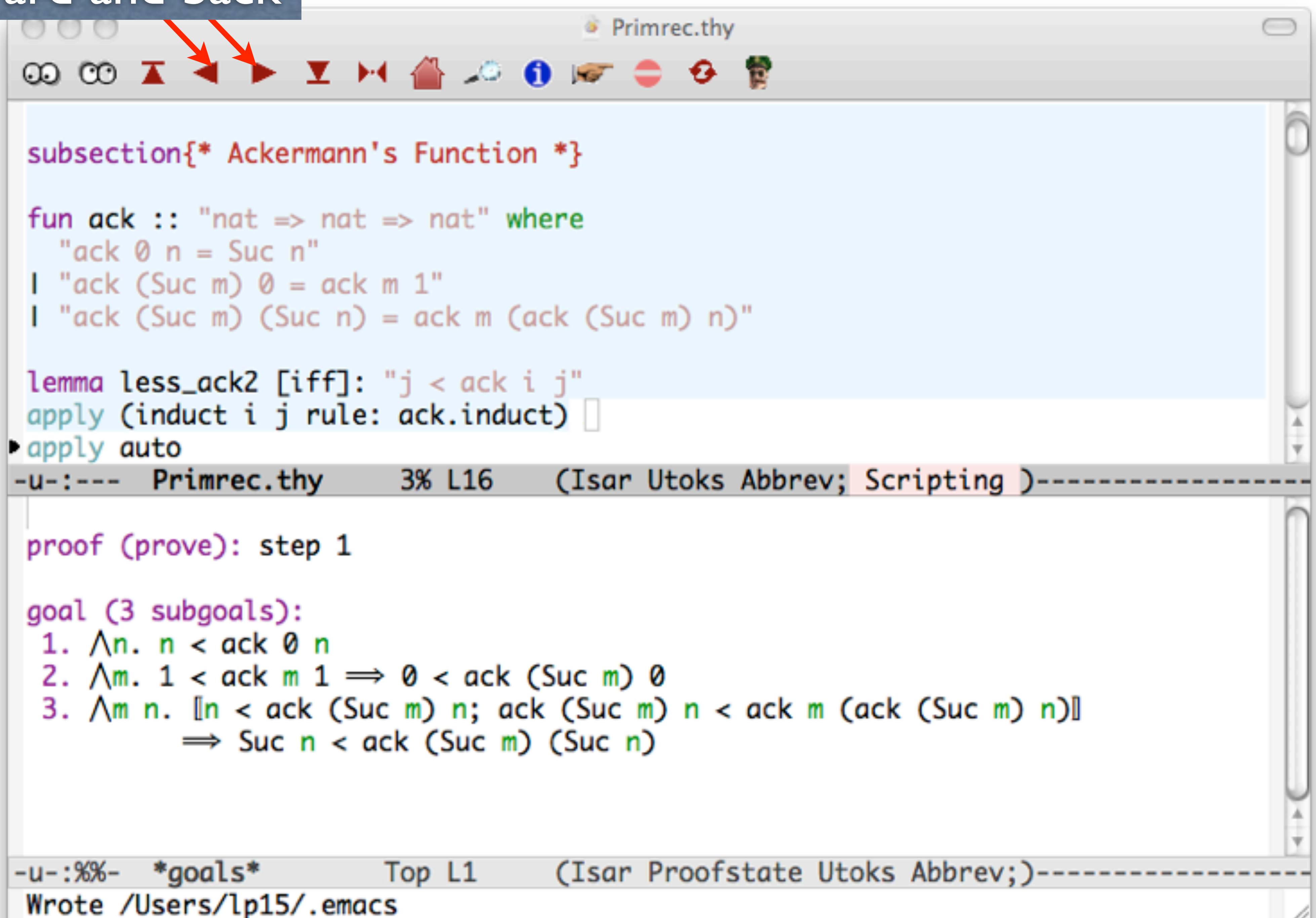

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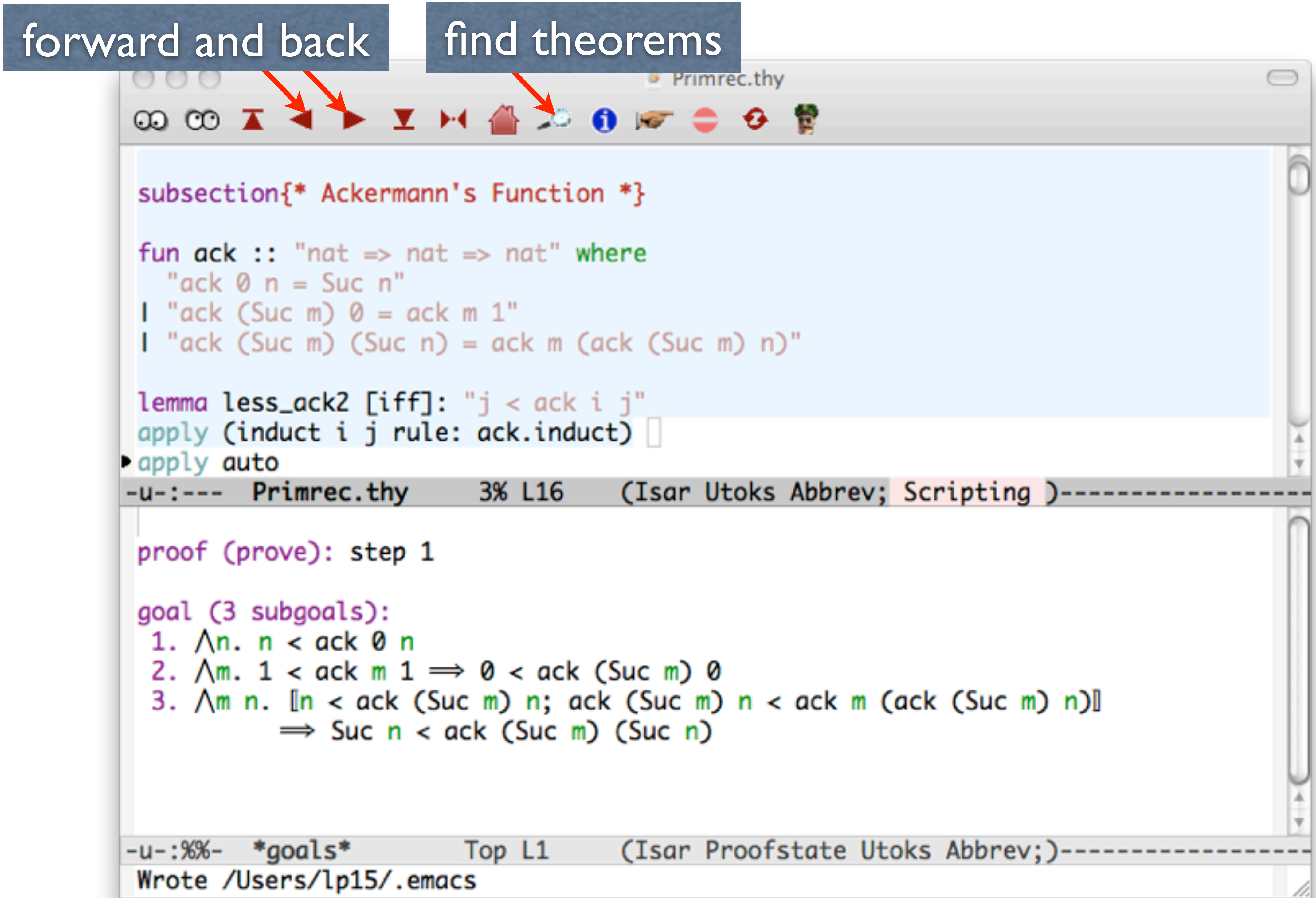

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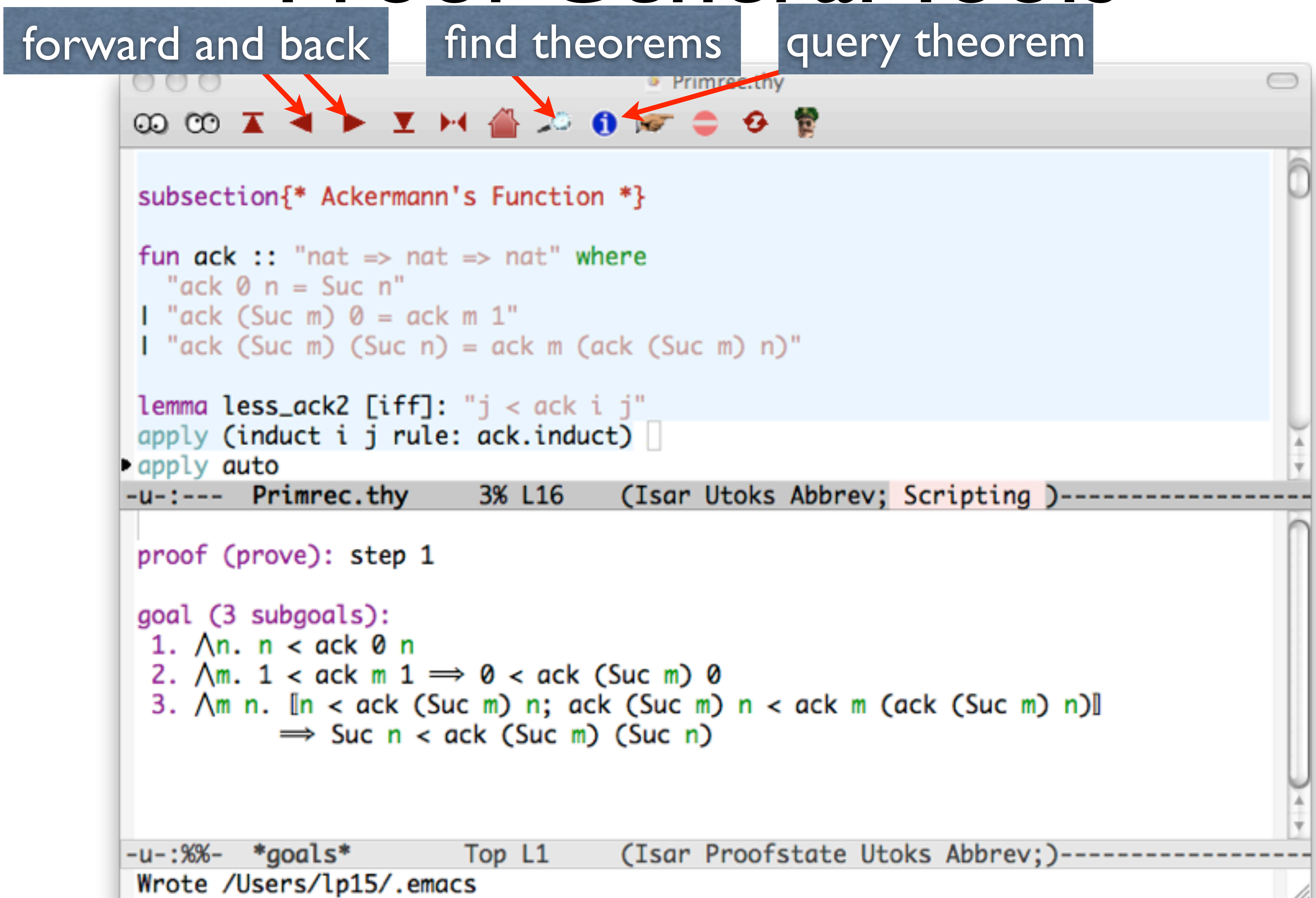

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